TUBI E CUSPIDI

(M,g) Riemanniana $\quad p \in M \quad \exp_p : T_pM \to M$

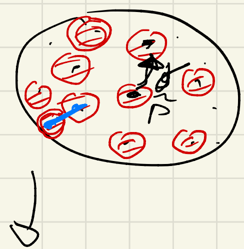
$$\mathrm{inj}_p := \sup\left\{R>0 \;\middle|\; \exp_p|_{B(0,r)} \text{ embedding}\right\}$$

$\mathrm{inj} : M \to (0,\infty]$ cont.

$$\mathrm{inj}(M) = \inf_{p\in M} \mathrm{inj}_p \qquad M \text{ cpt} \Rightarrow \mathrm{inj}(M) > 0$$

Def: $S \subseteq \mathbb{H}^n$ discreto $\quad d(S) = \inf\{d(x,y) \mid x\neq y,\ x,y\in S\}$

Prop: $\boxed{M = \mathbb{H}^n/\Gamma}$ $\qquad \mathrm{inj}_p = \frac{1}{2}\, d(\pi^{-1}(p)) \quad p\in M$



$\pi^{-1}(p) = \Gamma\tilde{p}$ discreto

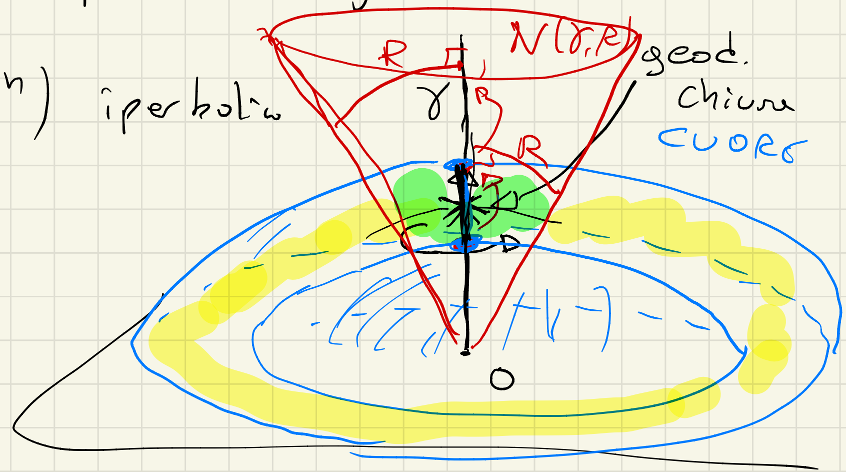
Cor: $\mathrm{inj}\,M = \frac{1}{2}\inf\{d(\gamma) \mid \gamma\in\Gamma,\ \gamma\neq id\}$

Cor: M cpt $\Rightarrow$ Γ non contiene parabolici

$\overset{\shortparallel}{\mathbb{H}^n/\Gamma}$

Γ cont. parab $\Rightarrow$ $\operatorname{inj} M = 0$

Def: TUBO: $\gamma \in \operatorname{Isom}(\mathbb{H}^n)$ iperbolica



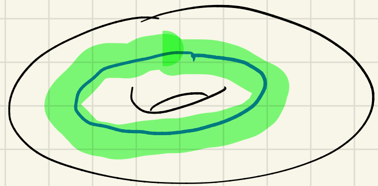
$M = \mathbb{H}^n/\Gamma$ $\quad \Gamma = \langle \gamma \rangle$

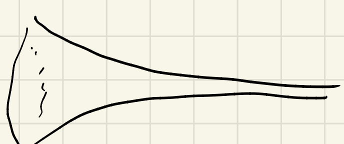
$d(\gamma^k) = k\, d(\gamma)$

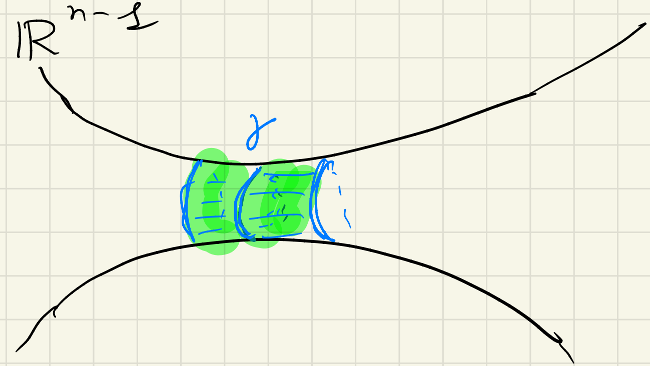
È un fibrato in piani su S^1

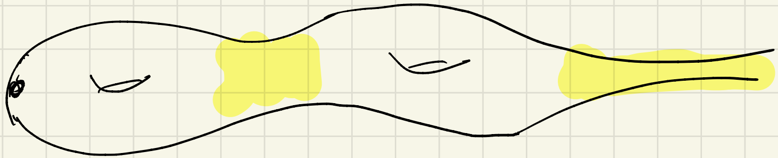
Se γ pres. orientaz. $\Rightarrow$ $M \cong S^1 \times \mathbb{R}^{n-1}$

dim 3: $S^1 \times \mathbb{R}^2$









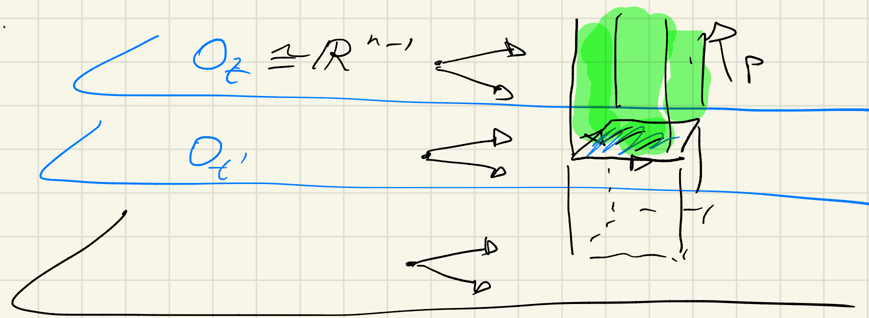
TUBO TRONCATO

$R>0$ $\quad \mathbb{H}^n/\Gamma$ $\quad N(\gamma, R)/\Gamma$

CUSPIDE :

$p \in \partial \mathbb{H}^n$ $\quad \Gamma = \{$ trasf. paraboliche che fissano $p \} \cup \{id\}$ discreto

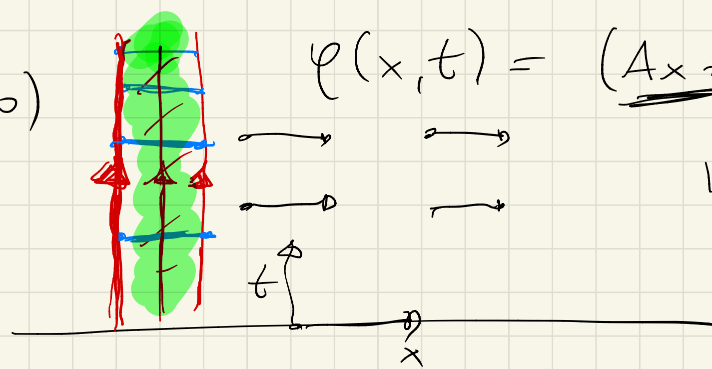
agire ~~lib.~~ & prop. disc.



$M = \mathbb{H}^n/\Gamma$ $\qquad H^n$ con $p=\infty$

$O_t/\Gamma = N$ EUCLIDEA PIATTA

$M \underset{diff}{\cong} N \times (0, \infty)$



$\varphi(x,t) = (\underline{Ax+b}, t)$

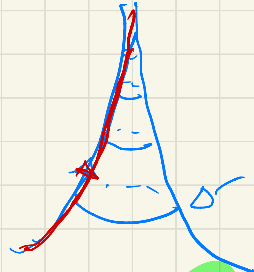
$\mathbb{H}^2$

$\varphi(x,t) = (x+1, t)$

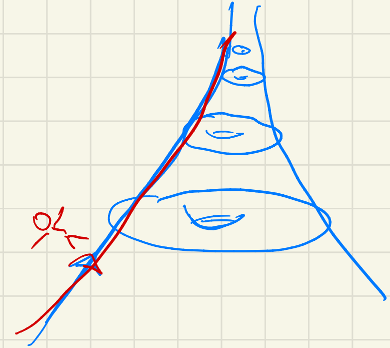
dim $n=3$

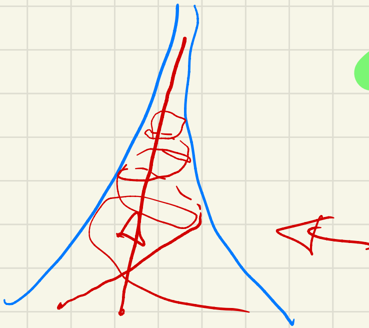
$u = e^t$

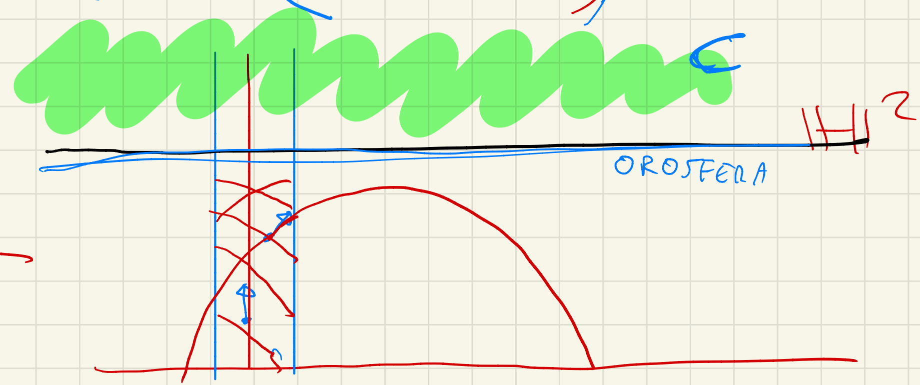
<u>Es</u>: Le geod. in una cuspide puntano dritte verso ∞ oppure tornano indietro



non sono geodetiche





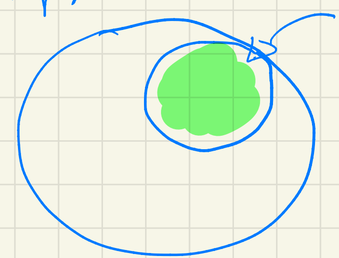


CUSPIDE TRONCATA:

Invece di $\mathbb{H}^n/\Gamma$ $\quad C/\Gamma = M$

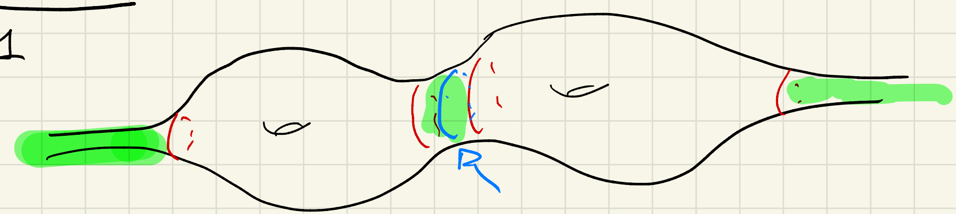
S^1

TORO



$M = N \times [t, \infty) \qquad \partial M = N$ non è geodetico

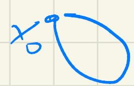
$$\mathrm{Vol}(M) = \frac{\mathrm{Vol}_{n-1}(\partial M)}{n-1}$$



GEODETICHE CHIUSE

G gruppo $\quad G^c$ sue classi di coniugio

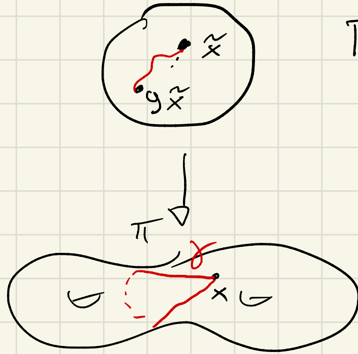
X sp. top. conn. p.a. $\qquad \pi_1(X, x_0) \longrightarrow [S^1, X]$



$[Y, X] = \{Y \to X \text{ cont}\}/_{\text{omot.}}$

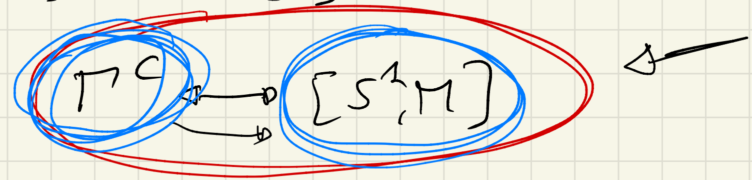
Ex: Indurre $\pi_1(X)^c \xrightarrow[1:1]{} [S^1, X]$

Se $M = \mathbb{H}^n/_\Gamma$



$$\Gamma^c = \mathrm{Aut}(\pi)^c \longleftrightarrow \pi_1(M,x)^c \longleftrightarrow [S^1,M]$$

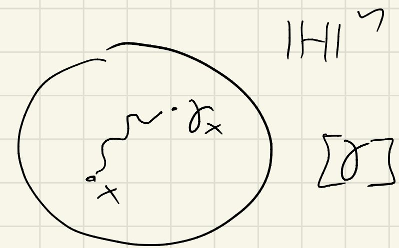
$$g \longmapsto [\gamma]$$



$[\gamma] \in \Gamma^c$ $\qquad \gamma \in \Gamma$

banale, ell., ip., par.

$M = \mathbb{H}^n / \Gamma$

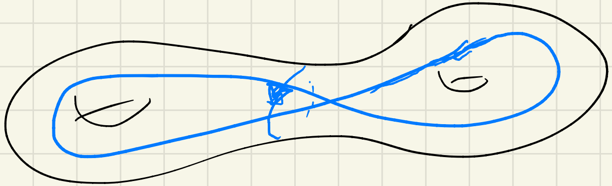


Prop: Se $[\gamma] \in \Gamma^c$ è ip, allora è rappresentata da un'unica geod. chiusa

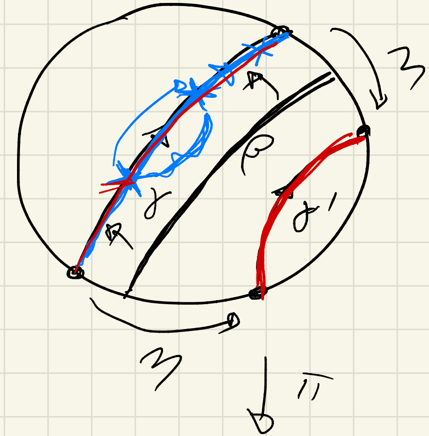
Se è ban par da nessuna geod. chiusa

Cor: $\{$geod. chiuse$\} \longleftrightarrow \{$classi di con. ip. in $\Gamma\}$

Def: (M,g) Una GEODETICA CHIUSA è geod. non banale periodica



dim: $[\gamma]$ ip.

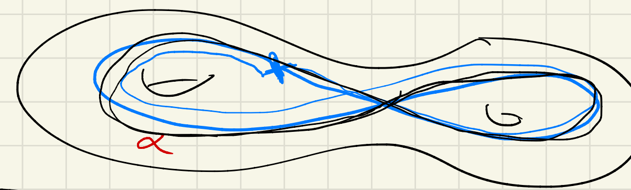


$\gamma \longmapsto \eta^{-1}\gamma\eta = \gamma'$

$\eta \in \Gamma$

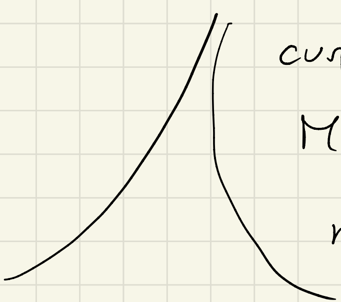
$[\gamma] \longrightarrow \alpha$

$[\gamma] \sim \alpha \longrightarrow \beta \subseteq \mathbb{H}^n$ invariante per γ



$\square$

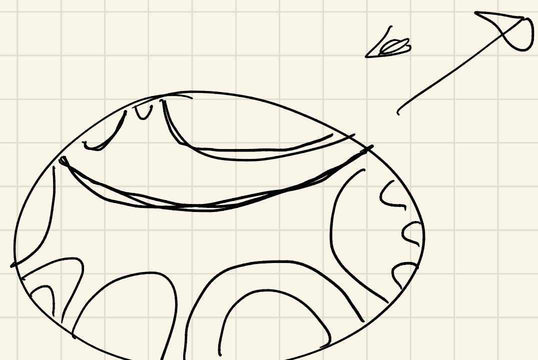
Oss: $[S^1, M]$



cuspide

$M = \mathbb{H}^n / \Gamma$

non ha geod chiuse



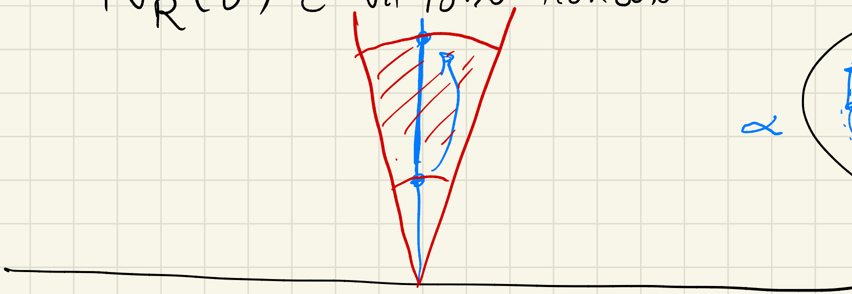
Cor: M cpt

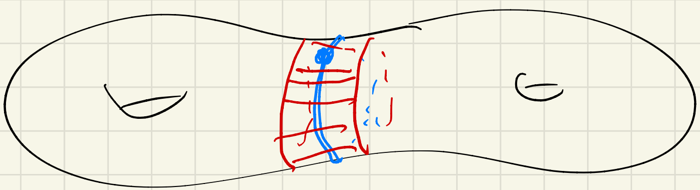
ogni el. non ban. in $[S^1, M]$ ha ! rappr. geod.

Oss: Se $\gamma \subseteq M = \mathbb{H}^2/\Gamma$ geod. chiusa semplice

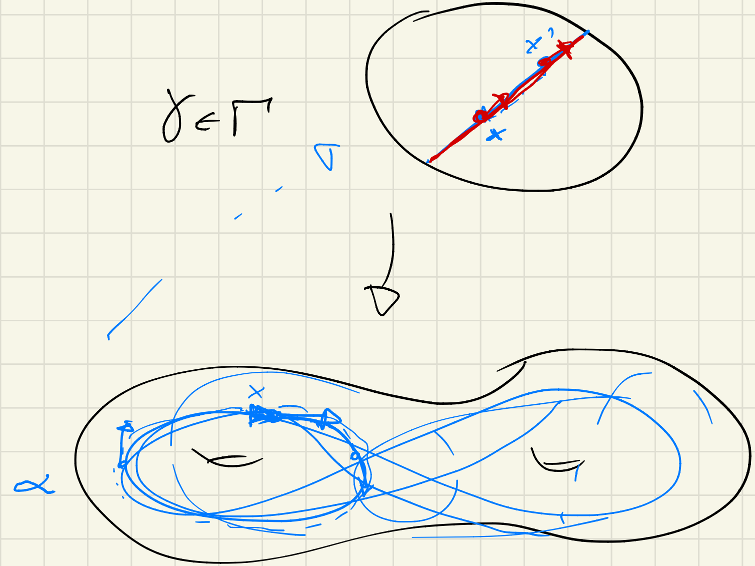
per $R>0$ piccolo

$N_R(\gamma)$ è un tubo troncato





$\gamma \in \Gamma$



LEMMA DI MARGULIS

$\mathbb{H}^n$

Lemma: $\varphi_1, \varphi_2 \in \mathrm{Isom}(\mathbb{H}^n)$ ip. o parab. commutano

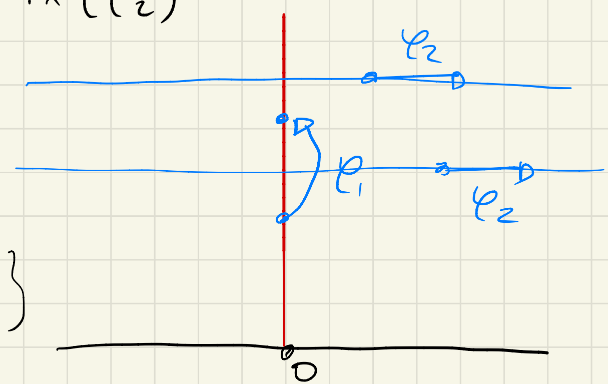
$\iff \mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$

Lemma: $\varphi_1, \varphi_2 \in \Gamma < \mathrm{Isom}(\mathbb{H}^n)$ ~~agli~~. $M = \mathbb{H}^n / \Gamma$ varietà

$\mathrm{Fix}(\varphi_1) \cap \mathrm{Fix}(\varphi_2) = \emptyset$, oppure

1) φ_1, φ_2 parab. con ~~la~~ $\mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$

2) $\exists\, \varphi \in \Gamma$ ip. $\quad \varphi_1 = \varphi^k, \ \varphi_2 = \varphi^h$



dim: a) φ_1 ip. $\quad \varphi_2$ par.

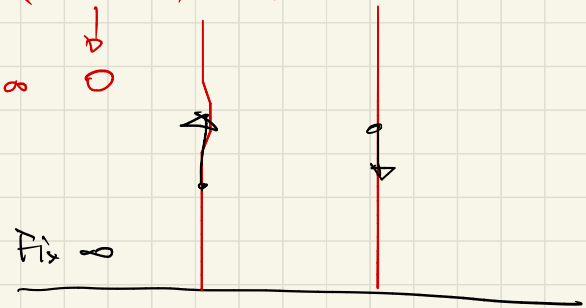
p.A.: $\mathrm{Fix}(\varphi_2) = \{\infty\}$ $\quad \mathrm{Fix}(\varphi_1) = \{0, \infty\}$

$\varphi_1(x,t) = \lambda(Ax, t)$

$\varphi_2(x,t) = (A'x + b, t)$

$\lambda < 1$ $\qquad \varphi_1^n \circ \varphi_2 \circ \varphi_1^{-n}(x,t) = (A^n A' A^{-n} x + \lambda^n A^n b,\ t)$

$$(0,1) \longmapsto (\lambda^n A^n b,\ 1) \xrightarrow{n\to\infty} 0$$



b) φ_1, φ_2 ip.

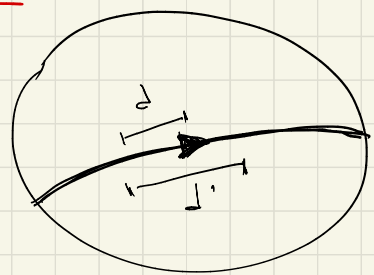
Fix $= \{a, \infty\}$ $\qquad$ Fix $= \{b, \infty\}$

$\underline{a = b}$

$[\varphi_1, \varphi_2]$

(a) ~~parabolica~~ con Fix ∞

oppure banale

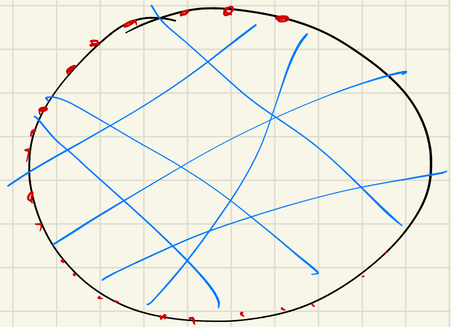


Γ discreto

<u>Cor</u>: $M = \mathbb{H}^n / \Gamma$ $\qquad$ Gli assi delle trasf. ip.

Due assi o sono incidenti o ultraparalleli

I punti in $\partial \mathbb{H}^n$ di quelle paraboliche

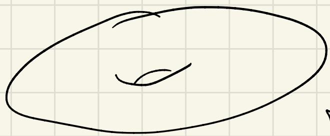


Cor: $M = \mathbb{H}^4 / \Gamma$ $\quad \mathbb{Z} \times \mathbb{Z} < \Gamma$

$\Downarrow$

è un gruppo di paraboliche che fissano stesso punto all'∞

Cor: M cpt $\Rightarrow \pi_1 M \not> \mathbb{Z} \times \mathbb{Z}$



NO METRICA IP.

GRUPPI NILPOTENTI

G gruppo $\qquad [h,k] = hkh^{-1}k^{-1}$

$H, K < G \qquad [H,K] = \langle [h,k] \mid h \in H, k \in K \rangle$

Prop: $H, K \triangleleft G \Rightarrow [H,K] < H \cap K \quad [H,K] \triangleleft G$

$G_0 = G \qquad G_n = [G_{n-1}, G] > G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$

$G > G_1 > G_2 > \dots$

Def: G NILPOTENTE se $G_n = \{e\}$ per qualche $\Rightarrow C(G)$ non banale

RISOLUBILE se $G^{(n)} = \{e\}$ $\Rightarrow$ ha sgr abel. normale

ABELIANO $\Rightarrow$ NILPOTENTE $\Rightarrow$ RISOLUBILE

$$Nil = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

Lemma: $G = \langle S \rangle$

Se $\exists n > 0$ t.c.

$\Rightarrow [a_1, [a_2, \dots [a_{n-1}, [a_n, b]] \dots] = e$

con $a_1, \dots, a_n, b \in S$

Allora G è nilpotente.

Lemma di Margulis:

G Lie $\exists U(e)$ t.c. $\forall \Gamma < G$ discreto generato da elementi di U

Γ è nilpotente

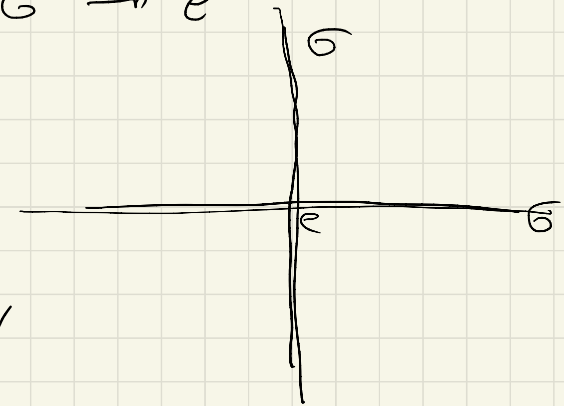
dim:

$[\,,\,] : G \times G \xrightarrow{\psi} G$

$(g, h) \longmapsto [g, h]$

$G \times \{e\} \to e$

$\{e\} \times G \to e$



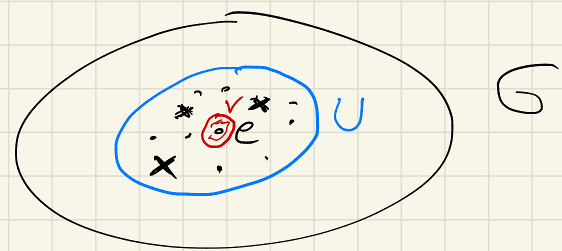
$\Rightarrow \quad d\psi_{(e,e)} = 0$

$\Rightarrow$ (in carta) $\quad \exists \; U_{(e)} \subseteq G$ t.c. $\forall \; V \subseteq U$

$\exists \; k > 0$ t.c. $\underline{[U \ldots, [U, [U, U]] \ldots] \subseteq V}$

$\forall \Gamma < G$ discreto gen. da el. di U

$\bar{e}$ nilp.



$V(e) \cap \Gamma = \{e\}$

Se $a_1, \ldots, a_n, b$ generatori di Γ in U

$[a_1, [\ldots\ ]] \in V \Rightarrow [a_1 \ldots\ ] = e$